Connectivity and distributions of three dimensional tilings

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Three Dimensional Tilings

**Domino** tilings of cubiculated Regions $R$:
- cubical complexes embedded as a finite polyhedron in $\mathbb{R}^N$
- connected oriented topological manifolds

**Dimer** covers of dual graph $R^*$

**Q:** Understand the space of tilings. What does a typical 3-dimensional tiling look like?

Focus: Connectivity by local moves. How and when can we move from one tiling to another?
**Connectivity - Local Moves**

**Flip**: Remove two adjacent parallel dominoes and place them back rotated within $2 \times 2 \times 1$ block.

**Theorem**: In two dimensions, any two tilings of a simply connected region are flip connected.

Not the case in 3d:

Two tilings of the $3 \times 3 \times 2$ box with no flips.
### Connectivity - Local Moves

<table>
<thead>
<tr>
<th>Box Size</th>
<th>Number of Tilings</th>
<th>Connected Components</th>
<th>Sizes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3 \times 3 \times 2$</td>
<td>229</td>
<td>3</td>
<td>227, 1, 1</td>
</tr>
<tr>
<td>$4 \times 4 \times 4$</td>
<td>5,051, 532, 105</td>
<td>93</td>
<td>4, 412, 646, 453</td>
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<td></td>
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<td>$2 \times 310, 185, 960$</td>
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<td></td>
<td>$2 \times 8, 237, 514$</td>
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<td></td>
<td>$2 \times 718, 308$</td>
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<td></td>
<td>$2 \times 283, 044$</td>
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<td></td>
<td>$6 \times 2, 576$</td>
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<td>$24 \times 618$</td>
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<td>$24 \times 236$</td>
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<td>$6 \times 4$</td>
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<tr>
<td></td>
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<td></td>
<td>$24 \times 1$</td>
</tr>
</tbody>
</table>
Connectivity - Local Moves

**Trit**: Remove and replace 3 dominoes, one parallel to each axis inside a $2 \times 2 \times 2$ box.

Question: Are tilings of three dimensional regions connected by flips and trits?
Question: Are all tilings of three dimensional regions connected by flips and trits?

In general, no.

Examples include tilings of:

- Cylinders: $\mathcal{D} \times [0, n]$
- Tori: $\mathbb{R}^3/\mathcal{L}$, $\mathcal{L} = 8\mathbb{Z}^3$.

Open:
- Boxes: $[0, L] \times [0, M] \times [0, N]$
Topological invariants

Two topological invariants: Flux, Twist.

Need notion of refinements.

Theorem (FKMS ’16)

For two tilings $t_0$ and $t_1$ of $R$:

- There exists a sequence of flips and trits connecting refinements of $t_0$ and $t_1$ if and only if $\text{Flux}(t_0) = \text{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of $t_0$ and $t_1$ if and only if $\text{Flux}(t_0) = \text{Flux}(t_1)$ and $\text{Twist}(t_0) = \text{Twist}(t_1)$. 
Refinement

• $R$ is refined by decomposing each cube into $5 \times 5 \times 5$ smaller cubes.

• $t$ is refined by decomposing each domino into $5 \times 5 \times 5$ smaller dominos, each parallel to the original.

Proposition

If $t_0$ and $t_1$ are connected by flips (resp. flips and trits) then their refinements are also connected by flips (resp. flips and trits).

(Converse false, examples in the $4 \times 4 \times 4$ box)
Flux - difference of tilings

For two tilings $t_0, t_1$

- $t_1 - t_0 := \text{union of tiles (with orientation of } t_0 \text{ reversed)}$.

Yields a system of cycles. (Ignore trivial 2-cycle.)

Homologically: $t_1 - t_0 \in \mathbb{Z}_1(R^*; \mathbb{Z})$
Topological Invariant - Flux

Fix a base tiling \( t_\oplus \).

\[
\text{Flux}(t) := [t - t_\oplus] \in H_1(R^*; \mathbb{Z})
\]

flip \( \leadsto \) boundary of a square

trit \( \leadsto \) boundary of 3 squares

**Proposition**

If \( t_0 \) and \( t_1 \) differ by flips and trits then \( \text{Flux}(t_0) = \text{Flux}(t_1) \).
A (discrete) Seifert surface for a pair $t_0, t_1$ is a connected embedded oriented topological surface $S$ (mapped continuously and injectively into $R^*$) with boundary $t_0 - t_1$.

**Proposition**

If $\text{Flux}(t_0) = \text{Flux}(t_1)$ then there exists a discrete Seifert surface in some refinement of the pair.
flux through a surface

\[ \varphi(v; t; S) = c(v) \cdot \begin{cases} 
+1, & \text{end above } S \\
0, & \text{end on } S \\
-1, & \text{end below } S 
\end{cases} \]

\[ \phi(t; S) = \sum_v \varphi(v; t; S) \]

c(v) is +1 if v is a black tile and −1 if v is a white tile.

**Theorem**

For S an embedded discrete surface with \( \partial(S) = \emptyset \),
if \( S = \partial(\text{manifold}) \) then \( \phi(t; S) = 0 \).
Flux vs. flux

• $\phi(t; S)$ really only depends on the homology class of the surface.

**Proposition**

If $\text{Flux}(t_0) = \text{Flux}(t_1)$ then $\phi(t_0; a) = \phi(t_1; a)$ for all $a \in H_2(R; \mathbb{Z})$.

Define the *modulus* of a tiling:

$$m := \mu(\text{Flux}(t)) := \gcd_{a \in H_2} \phi(t; a)$$

(Twist is well-defined up to the modulus.)
Twist

Fix a base tiling \( t_\oplus \):

\[
\text{Twist}(t) := \phi(t; t - t_\oplus) \in \mathbb{Z}/m\mathbb{Z}
\]

**Proposition**

If \( t_0 \leadsto \text{trit} \leadsto t_1 \) then

\[
\text{Flux}(t_0) = \text{Flux}(t_1) \text{ and } \text{Twist}(t_0) = \text{Twist}(t_1) \pm 1
\]

- Intuitively, the twist records how “twisted” a tiling is by trits.
- If \( \text{Flux}(t) = 0 \) then \( \text{Twist}(t) \in \mathbb{Z} \). (Boxes)
Main Theorem

Theorem

For two tilings $t_0$ and $t_1$ of $R$:

- There exists a sequence of flips and trits connecting refinements of $t_0$ and $t_1$ if and only if $\text{Flux}(t_0) = \text{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of $t_0$ and $t_1$ if and only if $\text{Flux}(t_0) = \text{Flux}(t_1)$ and $\text{Twist}(t_0) = \text{Twist}(t_1)$.

- Proof: height functions, winding forms.
Questions

Q: How often are refinements necessary?

Conjecture

For $N \in 2\mathbb{Z}$, consider the cubical torus $R = \mathbb{Z}^3/(N \cdot \mathbb{Z}^3)$. Select two tilings $t_0$ and $t_1$ of $R$ independently and uniformly at random.

- **A**: $t_0$ and $t_1$ are connected by flips;
- **B**: $\text{Flux}(t_0) = \text{Flux}(t_1)$ and $\text{Twist}(t_0) = \text{Twist}(t_1)$.

Then

$$\lim_{N \to \infty} \text{Prob}[A|B] = 1.$$

Open: Are boxes flip and trit connected?

Stronger: Region inside a box?
Distribution of Twist

Q: How is the twist distributed?

• Normally distributed?
• Giant component?

Data of the $4 \times 4 \times 4$ box.